This week

1. Section 3.1: tangents and the derivative at a point
2. Section 3.2: the derivative as a function
3. Section 3.4: velocity
The (angle of) **inclination** is the angle $\theta$ that $\ell$ makes with the $x$-axis.

- Turning counterclockwise means $\theta > 0$.
- Turning clockwise means $\theta < 0$. 

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**Inclination**

1.1

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UNIVERSITY OF TWENTE

Introduction to Mathematics and Modeling

Lecture 3: Differentiation
The **(angle of) inclination** is the angle $\theta$ that $\ell$ makes with the $x$-axis.

- The angle is measured from the positive $x$-axis to $\ell$. 
The (angle of) inclination is the angle $\theta$ that $\ell$ makes with the $x$-axis.

The angle is measured from the positive $x$-axis to $\ell$.

Turning counterclockwise means $\theta > 0$. 

Turning clockwise means $\theta < 0$. 

\[ y \]
\[ x \]
The (angle of) inclination is the angle $\theta$ that $\ell$ makes with the $x$-axis.

The angle is measured from the positive $x$-axis to $\ell$.

- Turning counterclockwise means $\theta > 0$.
- Turning clockwise means $\theta < 0$. 
The slope of \( \ell \) is defined as \( \tan \theta = \frac{\Delta y}{\Delta x} \).
The slope of a line

- The **slope of** $\ell$ is defined as $\tan \theta = \frac{\Delta y}{\Delta x}$.
- This holds for every choice $P_1$ and $P_2$, as long as $P_1 \neq P_2$. 
The slope of a line

The slope of \( \ell \) is 
\[
\tan \theta = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.
\]
The slope of a line

The slope of $\ell$ is $\tan \theta = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

This holds for every choice $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, as long as $P_1 \neq P_2$. 

Let $\ell$ be the line through $P = (x_0, y_0)$ with slope $m$, then for every point $(x, y) \neq P$ on $\ell$ we have

$$m = \frac{y - y_0}{x - x_0}.$$
Equation of a line through a point with given slope

Let \( \ell \) be the line through \( P = (x_0, y_0) \) with slope \( m \), then for every point \( (x, y) \neq P \) on \( \ell \) we have

\[
m = \frac{y - y_0}{x - x_0}.
\]

The equation of the line through \( P \) and with slope \( m \) is

\[
y = m(x - x_0) + y_0.
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Let \( \ell \) be the line through with slope \( m \) and with \( y \)-intercept \( b \), then \( \ell \) passes through \((0, b)\).
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The equation of \( \ell \) is

\[ y = m(x - 0) + b = mx + b. \]
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The equation of the line through \( P_1 \) and \( P_2 \) is

\[
y = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1.
\]
The horizontal line with $y$-intercept $b$ has slope 0 and therefore is described by the equation

$$y = b.$$
The horizontal line with $y$-intercept $b$ has slope 0 and therefore is described by the equation
\[ y = b. \]

The vertical line with $x$-intercept $a$ has slope $\infty$ and is described by the equation
\[ x = a. \]
Assignment: IMM1 - Tutorial 3.1.
The derivative of a function \( y = f(x) \)

We define the **derivative** \( f(x) \) at \( x_0 \) as

\[
f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.
\]

The number \( f'(x_0) \) can be interpreted as:

- the slope of the graph of \( y = f(x) \) at the point \((x_0, f(x_0))\);
- the slope of the tangent line to the graph of \( y = f(x) \) at the point \((x_0, f(x_0))\);
- the rate of change of \( f(x) \) at the point \( x_0 \).
Example: the derivative of $f(x) = x^2$ at 1

For $f(x) = x^2$ we have

<table>
<thead>
<tr>
<th>$h$</th>
<th>$1 + h$</th>
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\[
\begin{array}{|c|c|c|c|c|}
\hline
h & 1 + h & f(1) & f(1 + h) & \frac{f(1 + h) - f(1)}{h} \\
\hline
1 & 2 & 1 & 4 & 3 \\
.5 & 1.5 & 1 & 2.25 & 2.5 \\
\hline
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This suggests: when $h$ approaches 0, then $\frac{f(1 + h) - f(1)}{h}$ approaches 2.
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$$f(1) = 1^2 = 1$$
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(x + y)^2 = x^2 + 2xy + y^2
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\[
= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = 2.
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$$= \frac{2h + h^2}{h}$$

$$= 2 + h$$
Example: the derivative of \( f(x) = x^2 \) at 1

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f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = 2.
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Example: the tangent line of $f(x) = x^2$ at $(1, 1)$

- The tangent line has slope $f'(1) = 2$ and passes through $(1, f(1)) = (1, 1)$, hence the tangent line is described by the equation

$$y = 2 \cdot (x - 1) + 1 = 2x - 1.$$
Example: the derivative of $f(x) = x^2$ at $a$

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$(x + y)(x - y) = x^2 - y^2$
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$$= \frac{h}{h(\sqrt{a + h} + \sqrt{a})}$$  

$$= \frac{1}{\sqrt{a + h} + \sqrt{a}}.$$  

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f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \frac{1}{2\sqrt{a}}.
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Example: the derivative of \( f(x) = 1/x \) at \( a \neq 0 \)

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f(x) = \frac{1}{x} \quad (x \neq 0).
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$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = -\frac{1}{a^2}.$$
Assignment: IMM1 - Tutorial 3.2.
The derivative as a function

The **derivative of the function** \( f \) is the function \( f' \) whose value at \( x \) is

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

- The function \( f \) is **differentiable at** \( x \) if \( f'(x) \) exists.
The derivative of the function $f$ is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$  

- The function $f$ is **differentiable at** $x$ if $f'(x)$ exists.
- The process of calculating $f'$ is called **differentiation**.
The derivative as a function

The **derivative of the function** $f$ is the function $f'$ whose value at $x$ is

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- The function $f$ is **differentiable at** $x$ if $f'(x)$ exists.
- The process of calculating $f'$ is called **differentiation**.
- Alternative notations for the derivative are

\[
\frac{df}{dx}
\]

and

\[
\frac{d}{dx} f(x)
\]
Example: the derivative of $f(x) = x^2$

We already evaluated the derivative of $f$ at $a$:

$$f'(a) = 2a.$$ 

Replace $a$ by $x$: the derivative of $f$ is the function $f'(x) = 2x$. 

$$f(x) = x^2$$

$$f'(x) = 2x$$
Example: the derivative of $f(x) = \sqrt{x}$

The derivative of $f$ at $a$ is $f'(a) = \frac{1}{2\sqrt{a}}$.

Replace $a$ by $x$: the derivative of $f$ is the function $f'(x) = \frac{1}{2\sqrt{x}}$ ($x > 0$).
Example: the derivative of $f(x) = 1/x$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

The derivative of $f$ at $a$ is $f'(a) = -\frac{1}{a^2}$.

Replace $a$ by $x$: the derivative of $f$ is the function $f'(x) = -\frac{1}{x^2}$ \quad (x \neq 0).
Warning: derivatives are not always defined!

The graph of \( f(x) = \begin{cases} 
  x + 1 & \text{if } x \geq 0, \\
  -x & \text{if } x < 0
\end{cases} \) does not have a derivative at \( x = 0 \).

- A derivative does not exist at a point where the graph is discontinuous.
Warning: derivatives are not always defined!

\[ \frac{-2 - 1}{1 - 1} \]

The graph of \( y = f(x) = |x| \) does not have a derivative at \( x = 0 \).

- A derivative does not exist at a point where the graph has a sharp spike (called a cusp).
Warning: derivatives are not always defined!

\[ 3.7 \]

\[ -2 - 2 - 1 - 1 1 2 \]

\[ f \]

\[ f' \]

The graph of \( y = f(x) = \sqrt[3]{x} \) does not have a derivative at \( x = 0 \).

- A derivative does not exist at a point where the graph has a vertical tangent.
The function \( f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \) is differentiable at 0.

- Piecewise defined functions do not always pose problems.
The function $f(x) =$ \begin{align*}
\begin{cases}
  x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}
\end{align*}$ is differentiable at 0.
Assignment: IMM1 - Tutorial 3.3.
Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

- If the object moves from $s(t_A)$ to $s(t_B)$, the displacement is $s(t_B) - s(t_A)$.
- The average velocity over the interval $(t_A, t_B)$ is the displacement per elapsed time.
  $$\frac{s(t_B) - s(t_A)}{t_B - t_A}.$$
Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

- The **velocity at time** $t_A$ is the limit of the average velocity over the interval $(t_A, t_B)$ where $t_B$ approaches $t_A$:

$$v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}.$$
Consider a moving object and assume that we know the traveled distance as a function of time $s(t)$.

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  \[ v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}. \]
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$$v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}.$$
Velocity

\[ v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}. \]

Define \( h = t_B - t_A \), then

- \( t_B = t_A + h \) and
- “\( t_B \to t_A \)” is equivalent to “\( h \to 0 \).”

\[ v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A} \]

\[ = \lim_{h \to 0} \frac{s(t_A + h) - s(t_A)}{h} = s'(t_A). \]
Velocity

\[ v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A}. \]

Define \( h = t_B - t_A \), then

\[ t_B = t_A + h \] and

\[ "t_B \to t_A" \] is equivalent to \( "h \to 0" \).

\[ v(t_A) = \lim_{t_B \to t_A} \frac{s(t_B) - s(t_A)}{t_B - t_A} \]

\[ = \lim_{h \to 0} \frac{s(t_A + h) - s(t_A)}{h} = s'(t_A). \]

Velocity is the derivative of displacement.
Example: the motion of a rocket
Example: the motion of a rocket

Question: when did the rocket reach its highest point (apex)?
Example: the motion of a rocket

Question: when did the rocket reach its highest point (apex)?

Answer: at \( t \approx 8 \) seconds.
Example: the motion of a rocket

Question: for how many seconds did the engine burn?
Example: the motion of a rocket

Question: for how many seconds did the engine burn?

Answer: 2 seconds.
Example: the motion of a rocket

Question: when did the parachute open?

Answer: at $t = 10$ seconds.
Question: when did the parachute open?

Answer: at $t = 10$ seconds.
Example: the motion of a rocket

Question: what happens here?
Example: the motion of a rocket

<table>
<thead>
<tr>
<th>t (sec)</th>
<th>velocity (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<tr>
<td>4</td>
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<tr>
<td>6</td>
<td>100</td>
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<td>24</td>
<td>0</td>
</tr>
<tr>
<td>26</td>
<td>0</td>
</tr>
</tbody>
</table>

Question: what happens here?

Answer: after approximately 12 seconds the rocket reaches terminal velocity, which it keeps for about 8 seconds.
Example: the motion of a rocket

Question: what happens here?

Answer: the rocket hits the ground at \( t \approx 20 \) seconds.
Example: the motion of a rocket

Question: what happens here?

Answer: the rocket hits the ground at $t \approx 20$ seconds.
Example: the motion of a rocket

Question: what is the physical interpretation of the second derivative?
Example: the motion of a rocket

Question: what is the physical interpretation of the second derivative?

Answer: acceleration
The topic of the next lectures

It is tedious to compute derivatives by using the definition

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

How would you compute derivatives for these functions?

\[ f(x) = \sqrt{1 + \sin^2(x)} \]

\[ g(x) = (x^2 - 3x + 2)e^{-x^2} \]

\[ h(x) = \frac{x \tan(x)}{x^3 + x^2 + x + 1} \]

We need a set of rules to compute derivatives more efficiently.
Assignment: IMM1 - Tutorial 3.4.